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SHELL THEORY BASED ON INVARIANTS

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The precise theory is considered for finite strains of a three-dimensional body subordinate to the hypothesis of holding a normal element against a reference (central) surface. The first and second invariants of the strain tensor for a Green surface parallel to the reference surface are used as a measure of physical strains. It is shown that from the invariants of physical strains it is possible to determine any invariant characteristic of an elastic body: energy, stress tensor invariants, stress intensity, etc. A general definition is given for strain invariants of an arbitrary surface as components of the relative change in the square of a surface element. There is simplification of invariants with small strains and any distortions of thin shells. Expressions are obtained for the change in coefficients of the first and second quadratic forms of the central surface for small strains, and arbitrary and small displacements.

1. Geometry of a Three-Dimensional Body. We assume that $\mathbf{R}$ is radius vector of a three-dimensional body in the undeformed condition which is expressed in terms of reference surface radius vector $\mathbf{r}$ and the unit vector of the normal to the surface in the form $\mathbf{R} = \mathbf{r} + z\mathbf{n}$. In the general case $\mathbf{r}$ will be assumed to be independent of arbitrary curvilinear coordinates α_i. Coefficients of the first invariant form of the reference surface $a_{ij} = \mathbf{r}_{,i}\mathbf{r}_{,j}$, and for the surface z = const $A_{ij} = \mathbf{R}_{,i}\mathbf{R}_{,j}$. Here and subsequently i, j = 1, 2: indices after a comma signify differentiation with respect to α_i. The vector of the normal to surface z = const coincides with the vector of the normal to the base: $\mathbf{n} = (\mathbf{r}_{,1}\times\mathbf{r}_{,2})\, d_{aa}^{-1/2}$. For further convenience we adopt the following definition of the value $d_{\beta\gamma}$ which depends on the coefficients of any two quadratic forms β_{ij}, γ_{ij} ($d_{\beta\gamma} \neq d_{\gamma\beta}$):

$$d_{\beta\gamma} = \det\begin{Vmatrix}\beta_{11} & \beta_{12}\\ \gamma_{21} & \gamma_{22}\end{Vmatrix} = \beta_{11}\gamma_{22} - \beta_{12}\gamma_{21}.$$

Then $d_{aa} = a_{11}a_{22} - a_{12}^2$ is discriminant of quadratic form $a_{ij}d\alpha_i d\alpha_j$. The square of an element of area dF^2 of surface z = const has the form $dF^2 = d_{AA}d\alpha_1{}^2 d\alpha_2{}^2$. We assume that deformation of a three-dimensional body follows the hypothesis of holding a normal element against a reference surface [1]. In the deformed condition $\mathbf{R}^\vee = \mathbf{r}^\vee + z\mathbf{n}^\vee$, $a_{ij}^\vee = \mathbf{r}_{,i}^\vee\mathbf{r}_{,j}^\vee$, $A_{ij}^\vee = \mathbf{R}_{,i}^\vee\cdot\mathbf{R}_{,j}^\vee$, $\mathbf{n}^\vee = (\mathbf{r}_{,1}^\vee\times\mathbf{r}_{,2}^\vee)\, d_{aa}^{\vee\,-1/2}$, $dF^{\vee 2} = d_{AA}^\vee d\alpha_1^2 d\alpha_2^2$. Here for $d_{\beta\beta}$, where $\beta_{ij} = \gamma_{ij}{}^\vee$, we adopt the symbol $d_{\gamma\gamma}{}^\vee$.

2. Determination of Physical Strain Invariants. We consider surface z = const in the deformed condition. Assuming $A_{ij}{}^\vee = A_{ij} + 2E_{ij}$ and formulating the ratio $dF^{\vee 2}/dF^2$, we obtain

$$dF^{\vee 2}/dF^2 = 1 + 2I_E + 4I_{EE}; \tag{2.1}$$

$$I_E = (d_{AE} + d_{EA})/d_{AA}; \tag{2.2}$$

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$$I_{EE} = d_{EE}/d_{AA}; \tag{2.3}$$

$$E_{ij} = (1/2)\left(\mathbf{R}^{\vee}_{,i}\mathbf{R}^{\vee}_{,j} - \mathbf{R}_{,i}\mathbf{R}_{,j}\right). \tag{2.4}$$

We explain the meaning of I_E and I_{EE}. If we take as curvilinear coordinates α_i Cartesian coordinates x_i in a tangential plane at point 0 of surface z = const, then at this point $A_{ij} = \delta_{ij}$, $A_{ij}^{\vee} = \delta_{ij} + 2E_{ij}^*$ (δ_{ij} is Kronecker symbol). Here by E_{ij}^* we understand expression (2.4) calculated at point 0 in coordinate system x_i, E_{ij}^*, i.e., components of Green physical strains. These measures of strains are physical in the sense that in terms of them it is possible to express actual elongations and shifts without drawing on the metrics of the surface. Elongation of element dx_1 [2, 3] equals $[(1 + 2E_{11}^*)^{1/2} - 1]$. For small strains E_{ij}^* are actual elongations and shifts. Expressions (2.2) and (2.3) in coordinate system x_i take the form

$$I_E = E_{11}^* + E_{22}^*,\ I_{EE} = E_{11}^* E_{22}^* - E_{12}^{*2}.$$

According to the definition in elasticity theory [2, 4], I_E and I_{EE} are first and second invariants of the strain tensor (the other three strain tensor components for a three-dimensional body E_{13}^*, E_{23}^*, E_{33}^* are zero in view of the deformation hypothesis adopted in Part 1). As shown in elasticity theory, with affine deformation the ratio of areas $dF^{\vee}/dF$ of elementary figures constructed on vectors $\mathbf{R}_{,1}d\alpha_1$, $\mathbf{R}_{,2}d\alpha_2$, does not depend on figure shape and dimensions, and it is a characteristic of physical deformation at a point. In view of (2.1)

$$dF^{\vee}/dF = (1 + 2I_E + 4I_{EE})^{1/2}. \tag{2.5}$$

Thus, I_E and I_{EE} are connected with the relative change in area of an elementary figure. An approximate equality $dF^{\vee}/dF \simeq 1 + I_E$ is known which is obtained from (2.5) with small strains. In the case of an arbitrary set of coordinates α_i expressions E_{ij} (2.4) will not be physical strains. Equations (2.2) and (2.3) make it possible to calculate the invariants of physical strains without moving to calculation of individual components of E_{ij}^*. Methods for writing E_{ij}^* in terms of E_{ij} are studied in tensor analysis [1, 2].† In future we shall only consider the definition of invariant values.

3. Relationships for Elasticity and Energy. The elastic properties of an isotropic body may be represented by relationship between invariants of physical stresses and strains, and also expressions for the density of energy Π_V in a unit volume V of an elastic body. If it is assumed, as in classical shell theory, that at any surface z = const there are no stresses σ_{33}^* normal to the surface, then the stressed state at a point is governed by two invariants of physical stresses:

$$I_\sigma = \sigma_{11}^* + \sigma_{22}^*,\ I_{\sigma\sigma} = \sigma_{11}^*\sigma_{22}^* - \sigma_{12}^{*2}$$

(σ_{ij}^* are components of physical stresses in the coordinate system $\alpha_i = x_i$). In the case of a linearly elastic body there are the following expressions (taking account of $\sigma_{33}^* = 0$):

$$I_\sigma = 2(\lambda + \mu) I_E, \quad I_{\sigma\sigma} = \lambda(\lambda + 2\mu) I_E^2 + 4\mu^2 I_{EE},$$
$$\Pi_V = (1/2)\left[(\lambda + 2\mu) I_E^2 - 4\mu I_{EE}\right].$$

Here λ and μ are Lamé constants for a plane stressed state connected with elasticity modulus E and Poisson's ratio ν by the equalities $\lambda = \nu E(1 - \nu^2)$, $2\mu = E/(1 + \nu)$. As characteristics of the stressed state it is also possible to take other invariant values, for example, by using in strength theories the intensity of stresses σ (equivalent stress):

$$\sigma = (I_\sigma^2 - 3I_{\sigma\sigma})^{1/2}.$$

†It is noted that the relationship between E_{ij}^* and E_{ij} is ambiguous until no orientation is indicated for physical vectors $\mathbf{e}_i$ ($\mathbf{e}_i\mathbf{e}_j = \delta_{ij}$) with respect to reference vectors $\mathbf{R}_{,i}$ which may generate a new definition of components E_{ij}^*.

The total energy of an elastic body Π is described as the integral through the volume of energy density†

$$\Pi = \int_V \Pi_V \, dV, \quad dV = d_{AA}^{1/2} \, dz \, d\alpha_1 \, d\alpha_2.$$

Equilibrium equations are obtained from energy principles which use the variation of energy $\delta\Pi$. The main values of stresses $\sigma_1{}^*$ and $\sigma_2{}^*$ are found easily from invariants as the roots of a quadratic equation $\sigma^{*2} - I_\sigma \sigma^* + I_{\sigma\sigma} = 0$. The relationships provided show that the main problems of deformable body mechanics may be formulated in invariants.

4. Precise Relationships. According to (2.2), (2.3), I_E, I_{EE} at an arbitrary point are determined by the values A_{ij} and E_{ij} which are expressed in terms of characteristics of the reference surface. The corresponding relationships have the form

$$A_{ij} = a_{ij} + 2zb_{ij} + z^2 c_{ij}; \tag{4.1}$$

$$E_{ij} = \varepsilon_{ij} + z\varkappa_{ij} + (1/2)z^2 \nu_{ij}; \tag{4.2}$$

$$b_{ij} = -\mathbf{n}\mathbf{r}_{,ij}, \; c_{ij} = \mathbf{n}_{,i}\mathbf{n}_{,j}; \tag{4.3}$$

$$b_{ij}^{\vee} = -\mathbf{n}^{\vee}\mathbf{r}_{,ij}^{\vee}, \; c_{ij}^{\vee} = \mathbf{n}_{,i}^{\vee}\mathbf{n}_{,j}^{\vee}; \tag{4.4}$$

$$\varepsilon_{ij} = (1/2)\left(a_{ij}^{\vee} - a_{ij}\right), \; \varkappa_{ij} = b_{ij}^{\vee} - b_{ij}, \; \nu_{ij} = c_{ij}^{\vee} - c_{ij}. \tag{4.5}$$

Here $-b_{ij}(-b_{ij}{}^{\vee})$, $c_{ij}(c_{ij}{}^{\vee})$ are coefficients of the second and third quadratic forms‡ of an undeformed (deformed) reference surface; κ_{ij} and ν_{ij} are changes in coefficients of the second and third quadratic forms of the reference surface with deformation. Equations (4.1)-(4.5) make it possible to find I_E and I_{EE} in terms of the radius vector of the reference surface $\mathbf{r}^{\vee}$ in the deformed condition. These expressions together with (2.1)-(2.4) are correct with arbitrary strains and displacements, and they are precise for a three-dimensional body subordinate to the hypothesis of holding a normal element against a reference surface.

5. Approximate Relationships for I_E, I_{EE}, Π_F, Π. It is possible to obtain from Eqs. (2.2), (2.3), (4.1), and (4.2) approximate relationships for thin shells in which the central surface is taken as the reference surface. We provide a simple version based on the assumptions of technical thin shell theory with small strains for the central surface $\varepsilon_{ij}{}^*$. Ignoring the change in A_{ij} in a parallel surface and terms of the order of z^2 in (4.2), we have $A_{ij} \simeq a_{ij}$, $E_{ij} \simeq \varepsilon_{ij} + z\kappa_{ij}$. By determining I_E and I_{EE} from (2.2) and (2.3) without additional assumptions we obtain

$$\begin{gathered} I_E = I_\varepsilon + zI_\varkappa, \; I_{EE} = I_{\varepsilon\varepsilon} + zI_{\varepsilon\varkappa} + z^2 I_{\varkappa\varkappa}, \\ I_\varepsilon = (d_{a\varepsilon} + d_{\varepsilon a})/d_{aa}, \; I_{\varepsilon\varepsilon} = d_{\varepsilon\varepsilon}/d_{aa}, \\ I_\varkappa = (d_{a\varkappa} + d_{\varkappa a})/d_{aa}, \; I_{\varkappa\varkappa} = d_{\varkappa\varkappa}/d_{aa}, \\ I_{\varepsilon\varkappa} = I_{\varkappa\varepsilon} = (d_{\varepsilon\varkappa} + d_{\varkappa\varepsilon})/d_{aa}, \end{gathered} \tag{5.1}$$

where I_ε, $I_{\varepsilon\varepsilon}$ (I_κ, $I_{\kappa\kappa}$) are first and second invariants of the strain tensor (curvatures) of the central surface. Explicit expressions in terms of ε_{ij} and κ_{ij} have the form

$$\begin{gathered} I_\varepsilon = d_{aa}^{-1}(a_{22}\varepsilon_{11} + a_{11}\varepsilon_{22} - 2a_{12}\varepsilon_{12}), \\ I_{\varepsilon\varepsilon} = d_{aa}^{-1}\left(\varepsilon_{11}\varepsilon_{22} - \varepsilon_{12}^2\right), \\ I_\varkappa = d_{aa}^{-1}(a_{22}\varkappa_{11} + a_{11}\varkappa_{22} - 2a_{12}\varkappa_{12}), \\ I_{\varkappa\varkappa} = d_{aa}^{-1}\left(\varkappa_{11}\varkappa_{22} - \varkappa_{12}^2\right), \\ I_{\varepsilon\varkappa} = d_{aa}^{-1}(\varkappa_{22}\varepsilon_{11} + \varkappa_{11}\varepsilon_{22} - 2\varkappa_{12}\varepsilon_{12}). \end{gathered}$$

†A presentation of the energy of a three-dimensional elastic body in terms of a canonic tensor and its invariants is given in [5].

‡Coefficients of the third quadratic form are expressed in terms of coefficients of the first two forms [6].

For energy density Π_F in a unit area F of the central surface and total energy Π we find that

$$\Pi_F = \Pi_{F\varepsilon} + \Pi_{F\varkappa},\ c_h = h^3/12,\ \Pi_{F\varepsilon} = (1/2)\,h\left[(\lambda + 2\mu)\,I_\varepsilon^2 - 4\mu I_{\varepsilon\varepsilon}\right],$$
$$\Pi_{F\varkappa} = (1/2)\,c_h\left[(\lambda + 2\mu)\,I_\varkappa^2 - 4\mu I_{\varkappa\varkappa}\right],$$
$$\Pi = \int_F \Pi_F\,dF,\quad dF = d_{aa}^{1/2}\,d\alpha_1\,d\alpha_2$$

(h is shell thickness).

Similarly we determine invariants of the tensors of physical forces I_T and I_{TT} and moments I_M and I_{MM}, and also the intensity of forces T and moments M:

$$I_T = 2h\,(\lambda + \mu)\,I_\varepsilon,\ I_{TT} = h^2\left[\lambda\,(\lambda + 2\mu)\,I_\varepsilon^2 + 4\mu^2 I_{\varepsilon\varepsilon}\right],$$
$$I_M = 2c_h\,(\lambda + \mu)\,I_\varkappa,$$
$$I_{MM} = c_h^2\left[\lambda\,(\lambda + 2\mu)\,I_\varkappa^2 + 4\mu^2 I_{\varkappa\varkappa}\right],$$
$$T = (I_T - 3I_{TT})^{1/2},\ M = (I_M^2 - 3I_{MM})^{1/2}.$$

The main values of forces and moments are found as roots of quadratic equations

$$T^{*2} - I_T T^* + I_{TT} = 0,\ M^{*2} - I_M M^* + I_{MM} = 0.$$

In the case of reversion to zero of one of the main values of forces $T_2^* = 0$ and moments $M_2^* = 0$ (for example with cylindrical bending) $T = |T_1^*|$, $M = |M_1^*|$. With bending without strains for the central surface $d_{aa}{}^V = d_{aa}$. Gaussian curvature of the central surface, which is the second invariant I_{bb} of the curvature tensor, remains unchanged:

$$I_{bb} = d_{bb}^V/d_{aa}^V = d_{bb}/d_{aa}.$$

Then $I_{\kappa\kappa}$ (5.1) takes the form $I_{\kappa\kappa} = -I_{b\kappa} = -(d_{\kappa b} + d_{b\kappa})/d_{aa}$, i.e., $I_{\kappa\kappa}$, the same as for I_κ, is linear with respect to the κ_{ij} invariant. The relationships obtained in part 5 for the invariants of thin shells are correct with small strains of the central surface and any curvatures.

6. Approximate Relationships for $\mathbf{n}^V$, $\mathbf{v}$, ε_{ij}, $\varkappa_{ij}$. With small strains and arbitrary displacements ε_{ij} and κ_{ij} should be calculated by precise Eqs. (4.5). Simplification is only possible for the vector of normal $\mathbf{n}^V$ taking account of ε_{ij}, $\varkappa_{ij}$ (this equality is precise for an unstretched central surface, and in the general case $d_{aa}^V = d_{aa}(1 + 2I_\varepsilon + 4I_{\varepsilon\varepsilon})$):

$$\mathbf{n}^V \simeq (\mathbf{r}_{,1}^V \times \mathbf{r}_{,2}^V)\,d_{aa}^{-1/2},\ \varepsilon_{ij} = (1/2)\left(\mathbf{r}_{,i}^V\mathbf{r}_{,j}^V - \mathbf{r}_{,i}\mathbf{r}_{,j}\right),\ \varkappa_{ij} = -\left(\mathbf{n}^V\mathbf{r}_{,ij}^V - \mathbf{n}\mathbf{r}_{,ij}\right). \tag{6.1}$$

As follows from (6.1), with any displacements of a solid $\varepsilon_{ij} = \kappa_{ij} = 0$. Assuming $\mathbf{r}^V = \mathbf{r} + \mathbf{u}$, $\mathbf{n}^V = \mathbf{n} + \mathbf{v}$ ($\mathbf{u}, \mathbf{v}$ are vectors for displacements of the central surface and displacement of the normal) we obtain from (6.1) without additional assumptions geomechanically nonlinear relationships

$$\begin{gathered}\mathbf{v} = d_{aa}^{-1/2}\,(\mathbf{u}_{,1}\times\mathbf{r}_{,2} + \mathbf{r}_{,1}\times\mathbf{u}_{,2} + \mathbf{u}_{,1}\times\mathbf{u}_{,2}),\\ \varepsilon_{ij} = (1/2)(\mathbf{r}_{,i}\mathbf{u}_{,j} + \mathbf{r}_{,j}\mathbf{u}_{,i} + \mathbf{u}_{,i}\mathbf{u}_{,j}),\ \varkappa_{ij} = -(\mathbf{v}\mathbf{r}_{,ij} + \mathbf{n}\mathbf{u}_{,ij} + \mathbf{v}\mathbf{u}_{,ij}).\end{gathered} \tag{6.2}$$

Expressions (6.2) or their equivalents (6.1) determine all the values required in order to calculate the invariants in part 5. If the vector for displacements $\mathbf{u}$ is expanded over a local basis i the surface, then arbitrary $\mathbf{u}_{,i}$ are found using derivation equations for vector of the basis. With expansion of vector $\mathbf{u}$ with respect to unit vectors common for all of the surface of a Cartesian coordinate system derivation equations are not necessary since calculation of derivatives of the vector is reduced to determining normal derivatives of its components.

In the case of small displacements the relationships in Part 5 remain in force, and (6.2) may be linearized:

$$\mathbf{v} = d_{aa}^{-1/2}(\mathbf{u}_{,1} \times \mathbf{r}_{,2} + \mathbf{r}_{,1} \times \mathbf{u}_{,2}),$$

$$\varepsilon_{ij} = (1/2)(\mathbf{r}_{,i}\mathbf{u}_{,j} + \mathbf{r}_{,j}\mathbf{u}_{,i}),\ \varkappa_{ij} = -(\mathbf{v}\mathbf{r}_{,ij} + \mathbf{n}\mathbf{u}_{,ij}).$$

It is noted that it is also possible to obtain other approximate expressions for invariants of thin shells in the theory in question by expanding (2.2) and (2.3) into series with respect to powers of z.

7. The formulation and solution of boundary problems in nonlinear shell theory is connected with unknown difficulties. Direct variation methods based on using finite functions are currently most effective for shells of arbitrary shapes and boundaries [7]. Here a continuous system is brought to a system with discrete parameters. The version of the theory discussed gives a comparatively simple mathematical technique for work with curvilinear shell elements. A distinguishing feature of the theory is determination of values which are objective characteristics of strains which retain their numerical values at a given point of a body independent of the curvilinear coordinates adopted [8]. Equilibrium equations for a discrete system may be obtained from conditions of energy stability. In this case variation of energy is reduced to a finite number of variable parameters. If discrete parameters have a physical meaning (for example radius vectors and unit vectors of the normal at certain points of the surface [9]), then force boundary conditions are found naturally from variations in energy. Here both static and kinematic boundary conditions may be imposed. The structure of energy variations for discrete nonlinear shell models and also questions of algorithmization of computations have been discussed in [10, 11].

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